European Embedded Control Institute

M14 **09/05/2023 – 12/05/2023**	***Introduction to Nonlinear Systems and Control***

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Abstract of the course

This is a first course in nonlinear control with the target audience being engineers from multiple disciplines (electrical, mechanical, aerospace, chemical, etc.) and applied mathematicians.
The course is suitable for practicing engineers or graduate students who didn’t take such introductory course in their programs.
Prerequisites: Undergraduate-level knowledge of differential equations and control systems.

The course is designed around the text book:
H.K. Khalil, Nonlinear Control, Pearson Education, 2015

Outline

1. Introduction and second-order systems (phase portraits; multiple equilibrium points; limit cycles)
2. Stability of equilibrium points (basics concepts; linearization; Lyapunov’s method; the invariance principle; region of attraction; time-varying systems)
3. Perturbed systems; ultimate boundedness; input-to-state stability
4. Passivity and input-output stability
5. Stability of feedback systems (passivity theorems; the small-gain theorem; Circle & Popov criteria)
6. Normal and controller forms
7. Stabilization (concepts; linearization; feedback linearization; backstepping; passivity-based control)
8. Robust stabilization (sliding mode Control; Lyapunov redesign)
9. Observers (observers with linear-error dynamics; Extended Kalman Filter, high-gain observers)
10. Output feedback stabilization (linearization; passivity-based control; observer-based control; robust stabilization)
11. Tracking & regulation (feedback linearization; sliding mode Control; integral control)

INTRODUCTION TO NONLINEAR SYSTEMS & CONTROL

Lecture Plan
(90-minute lectures)

Lecture #	Topic	Time	Textbook
1	Introduction & Two-dimensional systems	Tuesday 14:00 – 15:30	Chapters 1 & 2
2	Stability of equilibrium points	Tuesday 16:00 – 17:30	Chapter 3
3	Stability of equilibrium points	Wednesday 9:00 – 10:30	Chapter 3 & Section 4.1
4	Perturbed systems	Wednesday 11:00 – 12:30	Chapter 4
5	Passivity & Input-output stability	Wednesday 14:00 – 15:30	Chapters 5 & 6
6	Stability of feedback systems	Wednesday 16:00 – 17:30	Chapter 7
7	Special Nonlinear Forms	Thursday 9:00 – 10:30	Chapter 8
8	State feedback stabilization	Thursday 11:00 – 12:30	Sections 9.1 – 9.4
9	State feedback stabilization	Thursday 14:00 – 15:30	Sections 9.5 – 9.7
10	Robust state feedback stabilization	Thursday 16:00 – 17:30	Section 10.1
11	Robust state feedback stabilization	Friday 9:00 – 10:30	Sections 10.1 – 10.3
12	Nonlinear Observers	Friday 11:00 – 12:30	Chapter 11
13	Output feedback stabilization	Friday 14:00 – 15:30	Chapter 12
14	Tracking & Regulation	Friday 16:00 – 17:30	Chapter 13

Textbook: H.K. Khalil, Nonlinear Control, Pearson Education, Upper Saddle River, New Jersey, 2015. The global edition has the same material but with some different exercises; see http://www.egr.msu.edu/~khalil/NonlinearControl/NoteontheGlobalEdition.pdf

EECI-M14 – May 2023
Introduction to Nonlinear Systems & Control
Exercises

All problems are 4 points each. The lowest three problems will be dropped, to bring the total to 100. To validate the module, the student needs to score 70/100. If the student does not achieve 70/100, he/she will receive comments on the solution and will be allowed to resubmit the solution of some problems. The solution has to be submitted as a pdf file and e-mailed to khalil@msu.edu by May 29, 2023. A corrected solution has to be submitted within one week from receiving comments on the earlier submission. The solution can be typed or handwritten.

1. Consider the system

$$\dot{x}_1 = -x_1 + 2x_2 + x_1x_2 + x_2^2, \qquad \dot{x}_2 = -x_1 - x_1^2 - x_1x_2$$

   (a) Find all equilibrium points and determine their types.

   (b) Construct and discuss the phase portrait.

2. Repeat the previous problem for the system

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = 2x_1 + 3x_2 + 2\ \mathrm{sat}(-3x_2)$$

3. For each of the following systems, determine whether the origin is stable, asymptotically stable or unstable.

   (1) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = -2\sin x_1 - 2x_2 - 2x_3$

   (2) $\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = 2\sin x_1 - 2x_2 - 2x_3$

4. Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\tanh(x_1 + x_2)$$

   (a) Show that

$$V(x) = \int_0^{x_1} \tanh(\sigma)\ d\sigma + \int_0^{x_1+x_2} \tanh(\sigma)\ d\sigma + x_2^2$$

   is positive definite for all x and radially unbounded.

   (b) Show that the origin is globally asymptotically stable.

5. An unforced mass-spring system with nonlinear viscous friction and nonlinear spring is modeled by

$$m\ddot{y} + b(1 + c|\dot{y}|)\dot{y} + g(y) = 0$$

   where $g(y) = k(1 - a^2y^2)y$, with $|ay| < 1$, for a softening spring, $g(y) = k(1 + a^2y^2)y$ for a hardening spring, and all constants are positive. Take the state variables as $x_1 = y$, $x_2 = \dot{y}$. Using an energy-type Lyapunov function, study the stability of the origin for each spring type.

6. Consider the system

$$\dot{x}_1 = -x_1 + x_3, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = -x_2 - x_2^3 - x_3$$

(a) Is the origin exponentially stable?

(b) Using $V(x) = bx_1^2 + 3x_2^2 + 2x_2x_3 + 2x_3^2 + x_2^4$, where $b > 0$ is to be chosen, show that the origin is globally asymptotically stable.

7. For each of the following systems determine whether the origin is uniformly stable, uniformly asymptotically stable, exponentially stable, or none of the above. In all cases, $g(t)$ is piecewise continuous and bounded.

(1) $\dot{x}_1 = -x_1^3 + g(t)x_2, \quad \dot{x}_2 = -g(t)x_1 - x_2$

(2) $\dot{x}_1 = g(t)x_2, \quad \dot{x}_2 = -g(t)x_1$

(3) $\dot{x}_1 = -g(t)x_1 + x_2, \quad \dot{x}_2 = x_1 - x_2$, $g(t) \geq 2 \ \forall \ t \geq 0$

8. Consider the system

$$\dot{x}_1 = -x_1 + \frac{2x_2}{1 + x_1^2}, \qquad \dot{x}_2 = \frac{-2x_1}{1 + x_1^2} - x_2 + 3a(x_1 + x_2)$$

(a) With $a = 0$, show that the origin is globally exponentially stable.

(b) Find an upper bound on $|a|$ such that the origin is globally exponentially stable.

9. Consider the system

$$\dot{x}_1 = -x_1 + \frac{x_2}{1 + x_1^2}, \qquad \dot{x}_2 = \frac{-x_1}{1 + x_1^2} - x_2 + a$$

Apply Theorem 4.5 to show that $x(t)$ is globally ultimately bounded and estimate the ultimate bound in terms of a.

10. For each of the following systems, investigate input-to-state stability.

(1) $\dot{x}_1 = -x_1 - x_1^3 + x_2, \qquad \dot{x}_2 = -x_1 - x_2 + u$

(2) $\dot{x}_1 = -x_1 - x_1^3 + x_2, \qquad \dot{x}_2 = -x_1 - x_2 + x_3, \qquad \dot{x}_3 = -x_3^3 + u$

11. Consider the system

$$M\dot{x} = -Kx + ah(x) + u, \qquad y = h(x)$$

where x, u, and y are n-dimensional vectors, M and K are positive definite symmetric matrices, h is locally Lipschitz, $h \in [0, K]$, and $\int_0^x h^T(\sigma)M \, d\sigma \geq 0$ for all x. Show that the system is passive when $a = 1$ and output strictly passive when $a < 1$.

12. Consider the system

$$\dot{x}_1 = -2x_1 + x_2, \qquad \dot{x}_2 = -2x_1 - \psi(x_1) + u, \qquad y = -x_1 + x_2$$

where ψ is a locally Lipschitz passive function.

(a) With $u = 0$, use $V(x) = \int_0^{x_1} \psi(\sigma)d\sigma + \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - x_1)^2$ to show that the origin is globally asymptotically stable.

(b) Show that the system is $\mathcal{L}_\infty$ stable.

13. Consider the system

$$\dot{x}_1 = -x_1 - x_2 + u, \qquad \dot{x}_2 = \psi_1(x_1) - \psi_2(x_2), \qquad y = x_1$$

where ψ_i belong to the sector $[\alpha_i, \beta_i]$ with $\beta_i > \alpha_i > 0$, for $i = 1, 2$.

(a) Represent the state equation as the feedback connection of two strictly passive dynamical systems.

(b) Show that the origin of the unforced system is globally exponentially stable.

(c) Let $u = -\psi_3(y) + r$, where ψ_3 belongs to the sector $[-\delta, \delta]$, with $\delta > 0$. Show that the mapping from r to y is finite-gain $\mathcal{L}_2$ stable for sufficiently small δ.

14. Consider the feedback connection on page 13 Lecture 6. Use the circle and Popov criteria to find a sector $[\alpha, \beta]$ for absolute stability when

$$G(s) = \frac{(s-1)}{(s+1)(s+2)}$$

15. Consider the system

$$\dot{x}_1 = x_1 + \frac{x_2}{1 + x_1^2}, \quad \dot{x}_2 = -x_2 + x_3, \quad \dot{x}_3 = x_1 x_2 + u$$

(a) If the output is x_2, find the relative degree of the system and determine if it is minimum phase.

(b) Repeat (a) if the output is x_3.

(c) Show that the system is feedback linearizable.

(d) Find a change of variables that transforms the system into the controller form and determine its domain of validity.

16. Consider the system

$$\dot{x}_1 = -x_1^3 + x_2 + 0.5 \sin x_2, \quad \dot{x}_2 = -x_1 - x_2 + x_3, \quad \dot{x}_3 = -x_3 + u$$

(a) Show that the system is feedback linearizable.

(b) Find a change of variables that transforms the system into the controller form and determine its domain of validity.

17. Design a globally stabilizing state feedback control for the system

$$\dot{x}_1 = -x_1 + \tanh(x_2), \qquad \dot{x}_2 = x_2 + x_3, \qquad \dot{x}_3 = u$$

18. Design a globally stabilizing state feedback control for the system

$$\dot{x}_1 = x_1 x_2, \qquad \dot{x}_2 = u$$

19. Design a globally stabilizing state feedback control for the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 + \text{sat}(u)$$

20. For the following system, use sliding mode control to design a locally Lipschitz, globally stabilizing state feedback controller. The constants θ_1 and θ_2 satisfy the bounds $0 \le \theta_1 \le a$, $0 < b \le \theta_2 \le c$, for known constants a, b, c. You need to verify that the controller will be stabilizing for $\mu < \mu^*$ for some $\mu^* > 0$ but you do not need to estimate μ^*.

$$\dot{x}_1 = x_2 + 2\sin x_1, \qquad \dot{x}_2 = \theta_1 x_1^2 + \theta_2 u$$

21. A normalized model of a pendulum whose suspension point is subjected to a time-varying bounded horizontal acceleration is given by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2 + cu + h(t)$$

where $b \in [0, 0.2]$, $c \in [0.5, 1.5]$, and $|h(t)| \le 2$. Design locally Lipschitz state feedback controller such that, for any initial state, $x(t) \in \{|x_1| \le 0.02,\ |x_2| \le 0.02\}$ for all $t \ge T$ for some finite time T.

22. A simplified model of an underwater vehicle in yaw is given by

$$\ddot{\psi} + a\dot{\psi}|\dot{\psi}| = u$$

where ψ is the heading angle, u is a normalized torque, and $a \in [0.5, 1.5]$ is a parameter. Using sliding mode control, design a locally Lipschitz state feedback controller to globally stabilize the system at the equilibrium point $(\psi = \psi_r,\ \dot{\psi} = 0)$.

23. Consider the van der Pol equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + (1 - x_1^2)x_2, \qquad y = x_1$$

(a) Design an Extended Kalman Filter.

(b) Design a high-gain observer with $\varepsilon = 0.01$.

(c) Compare the performance of the two observers using simulation.

24. Consider the pendulum equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 + a, \qquad y = x_1$$

with unknown positive constant a. Extend the state model with $x_3 = a$ and $\dot{x}_3 = 0$. Design an Extended Kalman Filter with $Q = R = P_0 = I$ and simulate it with $a = 1$, $x(0) = \text{col}(\pi/3, 0)$, and $\hat{x}(0) = \text{col}(0, 0, 0)$.

25. Design a globally stabilizing output feedback controller for the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 + \tanh(u), \qquad y = x_1$$

26. Design a stabilizing output feedback controller for the system

$$\dot{x}_1 = ax_1^2 + x_2, \qquad \dot{x}_2 = u, \qquad y = x_1, \qquad a \text{ is unknown with } |a| \le 1$$

such that the set $\{|x_1| \le 1,\ |x_2| \le 1\}$ is included in the region of attraction.

27. An electrostatic microactuator is modeled by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - 2\zeta x_2 + \frac{1}{3}x_3^2, \quad \dot{x}_3 = \frac{1}{T}\left[-(1 - x_1)x_3 + \frac{2}{3}u\right]$$

where ζ and T are positive constants. Consider a tracking problem where $r(t) = r_0 + (r_1 - r_0)q(t)$ and $q(t)$ is the output of the transfer function $1/(s+1)^3$ with a unit step input. The reference $r(t)$ is chosen to steer x_1 from equilibrium at $r_0 > 0$ to equilibrium at $r_1 > 0$.

(a) Design a locally Lipschitz state feedback controller using feedback linearization. Simulate the closed-loop system with $\zeta = 0.1$, $T = 0.2$, $r_0 = 0.1$, $r_1 = 0.5$, and $x(0) = \text{col}(0.1, 0, \sqrt{0.3})$.

(b) Design an observer to implement the controller of part (a) assuming you measure x_1 and x_3. Use simulation to compare the performance of the state and output feedback controllers.

28. A normalized model of a pendulum whose suspension point is subjected to a constant horizontal acceleration is given by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2 + cu + h$$

where $0 \le b \le 0.4$, $0.5 \le c \le 1.5$, and $|h| \le 1$.

(a) Design a locally Lipschitz state feedback controller to regulate x_1 to zero with zero steady-state error.

(b) Repeat (a) if you can only measure x_1.